Distance and similarity measures of Pythagorean fuzzy sets based on the Hausdorff metric with application to fuzzy TOPSIS

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Abstract
Pythagorean fuzzy sets (PFSs) were proposed by Yager in 2013 to treat imprecise and vague information in daily life more rigorously and efficiently with higher precision than intuitionistic fuzzy sets. In this paper, we construct new distance and similarity measures of PFSs based on the Hausdorff metric. We first develop a method to calculate a distance between PFSs based on the Hausdorff metric, along with proving several properties and theorems. We then consider a generalization of other distance measures, such as the Hamming distance, the Euclidean distance, and their normalized versions. On the basis of the proposed distances for PFSs, we give new similarity measures to compute the similarity degree of PFSs. Some examples related to pattern recognition and linguistic variables are used to validate the proposed distance and similarity measures. Finally, we apply the proposed methods to multicriteria decision-making by constructing a Pythagorean fuzzy Technique for Order Preference by Similarity to an Ideal Solution and then present a practical example to address an important issue related to social sector. Numerical results indicate that the proposed methods are reasonable and applicable and also that they are well suited in pattern recognition, linguistic variables, and multicriteria decision-making with PFSs.

KEYWORDS
distance, similarity measures, fuzzy sets, pythagorean fuzzy sets, hausdorff metric, multicriteria decision making
Fuzzy sets were developed by Zadeh in 1965 as an extension of crisp sets. Since then, fuzziness has been widely employed to handle a type of uncertainty that is different from probability (randomness). In real world, there exists much more knowledge with fuzziness that is vague, imprecise, and uncertainty in nature. Usually, for human judgment and reasoning involved in fuzzy information from genetically imprecise human belief, only humans can give reasonably true answers. Nevertheless, most systems based upon set theory and two-valued logic are not capable of providing answers to most questions. The main reason behind it is that most systems based upon set theory and two-valued logic are unable to manage with imprecise and vague information. However, systems for real-world problems are expected to be managed with imprecise and vague information and be able to give people more information. Fuzzy sets have generally been well used to provide solutions to most real-world problems.

To treat imprecise and vague information in daily life more rigorously, various applications and extensions of fuzzy sets were also proposed in the literature. A few of such theories as intuitionistic fuzzy sets (IFSs), interval-valued fuzzy set, type-2 fuzzy sets, fuzzy multisets, hesitant fuzzy sets. Fuzzy sets are based on a single membership function between 0 and 1. In the real-life setting, it may not be always true that nonmembership degree is equal to 1 minus membership value. Therefore, to get more purposeful reliability and applicability, Atanassov generalized fuzzy sets to IFSs that include both membership degree and nonmembership degree with a degree of non-determinacy as 1 minus membership degree minus nonmembership degree. An IFS $A$ in $X$ is given by Atanassov with a pair positive grade, denoted by $(\mu, \nu)$, that satisfies the condition $\mu + \nu \leq 1$, $\mu = \mu_a(x)$, $\nu = \nu_a(x)$, $x \in X$. Furthermore, Atanassov mentioned other types of IFSs, such as changing $\mu + \nu \leq 1$ with $\mu^2 + \nu^2 \leq 1$. It can be observed that $\mu^2 + \nu^2 \leq 1$ is an extension of the condition $\mu + \nu \leq 1$. Certainly, it can be continued toward increasing the power, but such a set with the condition $\mu^2 + \nu^2 \leq 1$ may not be useful and applicable.

Yager and Abbasov in 2013 started to make more advanced development on the condition $\mu^2 + \nu^2 \leq 1$, and introduced a class of Pythagorean fuzzy sets (PFSs) whose membership values are ordered pairs $(\mu, \nu)$, by fulfilling the condition of $\mu^2 + \nu^2 \leq 1$. Yager and Abbasov and Yager also provided aggregation operations on PFSs with applications to multicriterion decision-making (MCDM). Afterward, PFSs received more attention. According to Yager and Abbasov and Yager, the space of all Pythagorean membership values includes intuitionistic membership values. That is, PFSs are extensions of IFSs. For instance, the situation with the numbers $\mu = \sqrt{3}/2 = 0.86603$ (round off) and $\nu = 1/2 = 0.5$ is given as PFSs, but IFSs cannot be used. This is because $\mu + \nu = 1.36603 > 1$. Contrarily, we can use PFSs as $\mu^2 = (\sqrt{3}/2)^2 = 0.75$ and $\nu^2 = (1/2)^2 = 0.25$, because the condition $\mu^2 + \nu^2 = 1$ is fulfilled. Thus, PFSs can be used more widely than IFSs in tackling daily life problems with imprecision and uncertainty. Recently, there has been more research on PFSs in the literature. For example, Peng et al investigated the information measures and their application for PFSs. Biswas and Sarkar and Zeng proposed Pythagorean fuzzy group decision-making methods. Peng and Yang proposed the Pythagorean fuzzy TOPSIS approach to MCDM. Garg considered generalized Pythagorean fuzzy geometric aggregation operators.

Distance and similarity measures are important tools for determining the degrees of difference and similarity between two objects. Various distance and similarity measures about fuzzy sets and IFSs have been studied and applied in the literature. For example,
Liang and Shi proposed similarity measures on IFSs. Hung and Yang proposed similarity measures on IFSs based on Hausdorff distance. Xu et al. gave its application to clustering algorithm for IFSs, and so forth. In this paper, we propose new distance and similarity measures between PFSs based on the Hausdorff metric. We also consider a generalization of other distance measures, such as Hamming distance, Euclidean distance and their normalized versions. The organization of paper is as follows. In Section 2, we review some measures on IFSs and PFSs and present the concept of Hausdorff distance. In Section 3, we exhibit new methods to calculate the distance between PFSs based on Hausdorff distance. Furthermore, more generalizations of Hamming distance, Euclidean distance, and their normalized versions for PFSs are also considered. In Section 4, we use the distance to propose new similarity measures for PFSs. In Section 5, we use some examples related to pattern recognition and linguistic variables to validate the proposed methods. In Section 6, we construct a Pythagorean fuzzy Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) for multicriteria decision-making based on the proposed weighted Hausdorff distance that can lead us to select the best alternative among all alternatives to address an issue related to social sector. Finally, conclusions are stated in Section 7.

2 PRELIMINARIES

In this section, we give a brief review for IFSs, PFSs, distance, and Hausdorff metric.

2.1 Pythagorean fuzzy sets

We first review basic definitions of IFSs and PFSs.

**Definition 1** (Atanassov). An IFS $\tilde{M}$ in $X$ is defined by the form $\tilde{M} = \{(x, \mu_M(x), \nu_M(x)) | x \in X\}$ with the condition $0 \leq \mu_M(x) + \nu_M(x) \leq 1$, where the function $\mu_M(x) : X \rightarrow [0, 1]$ denotes the degree of membership of $x \in \tilde{M}$, and $\nu_M(x) : X \rightarrow [0, 1]$ denotes the degree of nonmembership of $x \in \tilde{M}$. For every $x \in X$, $\pi_M(x) = 1 - (\mu_M(x) + \nu_M(x))$ is called the intuitionistic fuzzy index of the element $x \in X$ to the IFS $M$ for representing the degree of uncertainty.

To model daily life problems carrying imprecision, uncertainty, and vagueness situations more precisely with higher accuracy, PFSs should be better than IFSs because the condition $\mu^2 + \nu^2 \leq 1$ is an extension of the condition $\mu + \nu \leq 1$ (see Yager and Abbasov).

**Definition 2** (Yager and Abbasov). A PFS $\tilde{P}$ in $X$ is given by $\tilde{P} = \{(x, \mu_P(x), \nu_P(x)) | x \in X\}$ with the condition $0 \leq \mu_P(x) + \nu_P(x) \leq 1$, where the function $\mu_P(x) : X \rightarrow [0, 1]$ represents the degree of membership of $x \in \tilde{P}$ and $\nu_P(x) : X \rightarrow [0, 1]$ represents the degree of nonmembership of $x \in \tilde{P}$. For every $x \in X$, $\pi_P(x) = \sqrt{1 - (\mu_P(x) + \nu_P(x))}$ is called Pythagorean fuzzy index of the element $x \in X$ to the PFS $\tilde{P}$ for representing the degree of uncertainty.

For example, let the universe of discourse be $X = \{x_1, x_2, x_3, x_4, x_5\}$. Then $\tilde{P} = \{\langle x_1, (0.9, 0.4) \rangle, \langle x_2, (0.8, 0.5) \rangle, \langle x_3, (0.7, 0.7) \rangle, \langle x_4, (0.6, 0.6) \rangle, \langle x_5, (0.4, 0.3) \rangle\}$ is a PFS in...
PFSs are more capable and efficient than IFSs to handle vagueness and uncertainty in real life settings. There are some situations as mentioned earlier that the IFS cannot handle, but PFSs can handle these situations amicably. Hence, to model some daily life problems by carrying imprecision, uncertainty and vagueness situations with more precise and high accuracy, PFSs are better than IFSs. Zhang and Xu\textsuperscript{25} considered Pythagorean fuzzy number (PFN) with $\gamma^n = ((\mu_x)^n, \sqrt{1 - (1 - \nu_x)^n})$, $\eta > 0$. We use it to define the PFS $\tilde{P}^n$ and then define the concentration and dilation operations as follows.

**Definition 3.** Let $\tilde{P} = \{ (x, \mu_{\tilde{P}}(x), \nu_{\tilde{P}}(x)) : x \in X \}$ be a PFS in $X$. For any positive number $n$, the PFS $\tilde{P}^n$ is defined as $\tilde{P}^n = \{ (x, (\mu_{\tilde{P}}(x))^n, \sqrt{1 - (1 - \nu_{\tilde{P}}(x))^n}) : x \in X \}, n > 0$.

It can be easily verified that, for any positive number $n$, $0 \leq [\mu_{\tilde{P}}(x)]^n + [\sqrt{1 - (1 - \nu_{\tilde{P}}(x))^n}] \leq 1$. Using Definition 3, the concentration and dilation of a PFS $\tilde{P}$ can be defined as $\text{CON}(\tilde{P}) = \{ (x, \mu_{\text{CON}(\tilde{P})}(x), \nu_{\text{CON}(\tilde{P})}(x)) : x \in X \}$ where $\mu_{\text{CON}(\tilde{P})}(x) = [\mu_{\tilde{P}}(x)]^2$ and $\nu_{\text{CON}(\tilde{P})}(x) = \sqrt{1 - [1 - \nu_{\tilde{P}}(x)]^2}$; and $\text{DIL}(\tilde{P}) = \{ (x, \mu_{\text{DIL}(\tilde{P})}(x), \nu_{\text{DIL}(\tilde{P})}(x)) : x \in X \}$ where $\mu_{\text{DIL}(\tilde{P})}(x) = [\mu_{\tilde{P}}(x)]^{1/2}$ and $\nu_{\text{DIL}(\tilde{P})}(x) = \sqrt{1 - [1 - \nu_{\tilde{P}}(x)]^{1/2}}$.

**Definition 4** (Yager and Abbasov\textsuperscript{9} and Yager\textsuperscript{10}). If $\tilde{P}$ and $\tilde{Q}$ are two PFSs in $X$, then

(i) $\tilde{P} \subseteq \tilde{Q}$ if and only if $\forall x \in X$, $\mu_{\tilde{P}}(x) \leq \mu_{\tilde{Q}}(x)$, and $\nu_{\tilde{P}}(x) \geq \nu_{\tilde{Q}}(x)$;
(ii) $\tilde{P} = \tilde{Q}$ if and only if $\forall x \in X$, $\mu_{\tilde{P}}(x) = \mu_{\tilde{Q}}(x)$, and $\nu_{\tilde{P}}(x) = \nu_{\tilde{Q}}(x)$;
(iii) $\tilde{P} \cup \tilde{Q} = \{ \max (\mu_{\tilde{P}}(x), \mu_{\tilde{Q}}(x)), \min (\nu_{\tilde{P}}(x), \nu_{\tilde{Q}}(x)) \}, \forall x \in X$;
(iv) $\tilde{P} \cap \tilde{Q} = \{ \min (\mu_{\tilde{P}}(x), \mu_{\tilde{Q}}(x)), \max (\nu_{\tilde{P}}(x), \nu_{\tilde{Q}}(x)) \}, \forall x \in X$.

### 2.2 Distance measures

Dissimilarity between two objects corresponding to fuzzy sets had been widely used in many applications to differentiate the two objects. Particularly, well-known distances, such as Hamming distance, Euclidean distance, and Hausdorff metric, are broadly considered for fuzzy sets. Let the universe of discourse be $X = \{x_1, ..., x_n\}$, and let $\tilde{A}$ and $\tilde{B}$ be any two fuzzy sets of $X$ with the membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$, respectively. The following distances between $\tilde{A}$ and $\tilde{B}$ were defined (see Kacpzyk\textsuperscript{26})�:

(i) Hamming distance

\[
d_{H}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|,
\]

(ii) Normalized Hamming distance

\[
d_{\text{norm}}(\tilde{A}, \tilde{B}) = \frac{1}{n} \sum_{i=1}^{n} |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|,
\]
(iii) Euclidean distance
\[ e_l(\tilde{A}, \tilde{B}) = \sqrt{\sum_{i=1}^{n} (\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i))^2}, \quad (3) \]

(iv) Normalized Euclidean distance
\[ e_{nlf}(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i))^2}. \quad (4) \]

Atanassov\(^2\) introduced nonmembership degree and gave direct generalizations of distances (1)–(4) for IFSs. Let \( \tilde{A} \) and \( \tilde{B} \) be any two IFSs; the following distances for the IFSs \( \tilde{A} \) and \( \tilde{B} \) are given:

(i) Hamming distance
\[ d_{lf}(\tilde{A}, \tilde{B}) = \frac{1}{2} \sum_{i=1}^{n} [ |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)| + |\nu_{\tilde{A}}(x_i) - \nu_{\tilde{B}}(x_i)|], \quad (5) \]

(ii) Normalized Hamming distance
\[ d_{nlf}(\tilde{A}, \tilde{B}) = \frac{1}{2n} \sum_{i=1}^{n} [ |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)| + |\nu_{\tilde{A}}(x_i) - \nu_{\tilde{B}}(x_i)|], \quad (6) \]

(iii) Euclidean distance
\[ e_{lf}(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{2} \sum_{i=1}^{n} [(\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i))^2 + (\nu_{\tilde{A}}(x_i) - \nu_{\tilde{B}}(x_i))^2]}, \quad (7) \]

(iv) Normalized Euclidean distance
\[ e_{nlf}(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} [(\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i))^2 + (\nu_{\tilde{A}}(x_i) - \nu_{\tilde{B}}(x_i))^2]}]. \quad (8) \]

Szmidt and Kacprzyk\(^1\) modified these distances (5)–(8) by considering all three parameters of IFSs viz. membership degree \( \mu(x) \), nonmembership degree \( \nu(x) \), and intuitionistic fuzzy index \( \pi(x) \) as follows:

(i) Hamming distance
\[ d_{lf}(\tilde{A}, \tilde{B}) = \frac{1}{2} \sum_{i=1}^{n} [ |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)| + |\nu_{\tilde{A}}(x_i) - \nu_{\tilde{B}}(x_i)| + |\pi_{\tilde{A}}(x_i) - \pi_{\tilde{B}}(x_i)|], \quad (9) \]
(ii) Normalized Hamming distance

\[
\tilde{d}_{if}(\tilde{A}, \tilde{B}) = \frac{1}{2n} \sum_{i=1}^{n} [|\mu_{A}(x_i) - \mu_{B}(x_i)| + |\nu_{A}(x_i) - \nu_{B}(x_i)| + |\pi_{A}(x_i) - \pi_{B}(x_i)|],
\]

(iii) Euclidean distance

\[
\tilde{e}_{if}(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{2} \sum_{i=1}^{n} [(\mu_{A}(x_i) - \mu_{B}(x_i))^2 + (\nu_{A}(x_i) - \nu_{B}(x_i))^2 + (\pi_{A}(x_i) - \pi_{B}(x_i))^2]},
\]

(iv) Normalized Euclidean distance

\[
\tilde{e}_{nif}(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} [(\mu_{A}(x_i) - \mu_{B}(x_i))^2 + (\nu_{A}(x_i) - \nu_{B}(x_i))^2 + (\pi_{A}(x_i) - \pi_{B}(x_i))^2]}.
\]

Yager and Abbasov\textsuperscript{9} and Yager\textsuperscript{10} generalized IFSs to PFSs that contain IFSs and those values in which the sum of membership \(\mu(x)\) and nonmembership \(\nu(x)\) is larger than 1. They proposed PFSs with the condition \(\mu^2(x) + \nu^2(x) \leq 1\). Thus, we extend Hamming distance, Euclidean distance, and their normalized counterparts to PFSs. Let \(\tilde{P}\) and \(\tilde{Q}\) be two PFSs of the set \(X\), we define the following distances for PFSs:

(i) Hamming distance

\[
h_{if}(\tilde{P}, \tilde{Q}) = \frac{1}{2} \sum_{i=1}^{n} [|\mu_{P}(x_i) - \mu_{Q}(x_i)| + |\nu_{P}(x_i) - \nu_{Q}(x_i)|],
\]

(ii) Normalized Hamming distance

\[
h_{nif}(\tilde{P}, \tilde{Q}) = \frac{1}{2n} \sum_{i=1}^{n} [|\mu_{P}(x_i) - \mu_{Q}(x_i)| + |\nu_{P}(x_i) - \nu_{Q}(x_i)|],
\]

(iii) Euclidean distance

\[
e_{if}(\tilde{P}, \tilde{Q}) = \sqrt{\frac{1}{2} \sum_{i=1}^{n} [(\mu_{P}(x_i) - \mu_{Q}(x_i))^2 + (\nu_{P}(x_i) - \nu_{Q}(x_i))^2]},
\]

(iv) Normalized Euclidean distance

\[
e_{nif}(\tilde{P}, \tilde{Q}) = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} [(\mu_{P}(x_i) - \mu_{Q}(x_i))^2 + (\nu_{P}(x_i) - \nu_{Q}(x_i))^2]}.
\]
While considering all the three parameters of PFSs viz. membership degree $\mu(x)$, nonmembership degree $\nu(x)$, and Pythagorean fuzzy index $\pi(x)$, we can also define the following distances for PFSs:

(i) Hamming distance

$$\widetilde{h}_p(\tilde{P}, \tilde{Q}) = \frac{1}{2} \sum_{i=1}^{n} \{|\mu^2_i(x_i) - \mu^2_i(x_i)| + |\nu^2_i(x_i) - \nu^2_i(x_i)| + |\pi^2_i(x_i) - \pi^2_i(x_i)|\}, \quad (17)$$

(ii) Normalized Hamming distance

$$\widetilde{h}_{np}(\tilde{P}, \tilde{Q}) = \frac{1}{2n} \sum_{i=1}^{n} \{|\mu^2_i(x_i) - \mu^2_i(x_i)| + |\nu^2_i(x_i) - \nu^2_i(x_i)| + |\pi^2_i(x_i) - \pi^2_i(x_i)|\}, \quad (18)$$

(iii) Euclidean distance

$$\widetilde{e}_p(\tilde{P}, \tilde{Q}) = \frac{1}{2} \sum_{i=1}^{n} \{(\mu^2_i(x_i) - \mu^2_i(x_i))^2 + (\nu^2_i(x_i) - \nu^2_i(x_i))^2 + (\pi^2_i(x_i) - \pi^2_i(x_i))^2\}, \quad (19)$$

(iv) Normalized Euclidean distance

$$\widetilde{e}_{np}(\tilde{P}, \tilde{Q}) = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} \{(\mu^2_i(x_i) - \mu^2_i(x_i))^2 + (\nu^2_i(x_i) - \nu^2_i(x_i))^2 + (\pi^2_i(x_i) - \pi^2_i(x_i))^2\}}, \quad (20)$$

2.3 | Hausdorff metric

Literally, the Hausdorff metric is a measure of how far two nonempty closed and bounded (compact) subsets $P$ and $Q$ in a Banach space $S$ look like each other with respect to their position in the metric space. The Hausdorff metric is defined as the maximum distance of a set to the nearest point in the other set (see Nalder, Huttenlocher et al, Oslon, and Rote). Let $d(x, y)$ be a metric for any two subsets $P$ and $Q$ in a Banach space $S$. Then, the forward distance and the backward distance are defined as follows:

$$h(P, Q) = \max_{x \in P} \left\{ \min_{y \in Q} (\|x - y\|) \right\} \quad \text{(forward distance)}$$

$$h(Q, P) = \max_{y \in Q} \left\{ \min_{x \in P} (\|x - y\|) \right\} \quad \text{(backward distance)}$$

The Hausdorff metric is oriented, in other words, asymmetric as well, which means that most of the time, the forward distance $h(P, Q)$ is not equal to the backward distance $h(Q, P)$. The Hausdorff metric is defined as follows:

$$H(P, Q) = \max\{h(P, Q), h(Q, P)\} \quad (21)$$
If \( S = \mathbb{R} \), \( P = [p_1, p_2] \), and \( Q = [q_1, q_2] \) are intervals, then Equation (21) shrinks to be

\[
H(P, Q) = \max\{|p_1 - q_1|, |p_2 - q_2|\}. \tag{22}
\]

Distance and similarity measures play a vital role to differentiate two sets or objects. Distance and similarity have seen a lot of applications.\textsuperscript{31,32} Therefore, we give novel distance and similarity measures between two PFSs based on the Hausdorff metric in the next section.

### 3 | DISTANCE FOR PFSs BASED ON HAUSDORFF METRIC

Distance is important in many fields of applications. In this section, we first define a distance between two PFSs based on the Hausdorff metric. Hausdorff metric for intervals could be extended to PFSs. Let \( \tilde{P} \) and \( \tilde{Q} \) be any two PFSs on the finite universe \( X = \{x_1, x_2, ..., x_n\} \). We set the two intervals \( I_{\tilde{P}}(x_i) \) and \( I_{\tilde{Q}}(x_i) \) as subintervals in \([0, 1]\) with

\[
I_{\tilde{P}}(x_i) = [\mu_{\tilde{P}}(x_i), 1 - \nu_{\tilde{P}}(x_i)],
\]

and

\[
I_{\tilde{Q}}(x_i) = [\mu_{\tilde{Q}}(x_i), 1 - \nu_{\tilde{Q}}(x_i)],
\]

for \( i = 1, 2, ..., n \). Let \( H(I_{\tilde{P}}(x_i), I_{\tilde{Q}}(x_i)) \) be the Hausdorff distance between \( I_{\tilde{P}}(x_i) \) and \( I_{\tilde{Q}}(x_i) \). We then define the Hausdorff distance \( d_{H}(\tilde{P}, \tilde{Q}) \) between two PFSs \( \tilde{P} \) and \( \tilde{Q} \) as

\[
d_{H}(\tilde{P}, \tilde{Q}) = H(I_{\tilde{P}}(x_i), I_{\tilde{Q}}(x_i)) = \max\{|\mu_{\tilde{P}}(x_i) - \mu_{\tilde{Q}}(x_i)|, |1 - \nu_{\tilde{P}}(x_i) - (1 - \nu_{\tilde{Q}}(x_i))|\},
\]

and we have

\[
d_{H}(\tilde{P}, \tilde{Q}) = \max\{|\mu_{\tilde{P}}(x_i) - \mu_{\tilde{Q}}(x_i)|, |\nu_{\tilde{P}}(x_i) - \nu_{\tilde{Q}}(x_i)|\}.
\]

Thus, we propose a new Hausdorff distance \( d_{H}(\tilde{P}, \tilde{Q}) \) between PFSs \( \tilde{P} \) and \( \tilde{Q} \) with the normalized Hausdorff distance as

\[
d_{H}(\tilde{P}, \tilde{Q}) = \frac{1}{n} \sum_{i=1}^{n} H(I_{\tilde{P}}(x_i), I_{\tilde{Q}}(x_i)) = \frac{1}{n} \sum_{i=1}^{n} \max\{|\mu_{\tilde{P}}(x_i) - \mu_{\tilde{Q}}(x_i)|, |\nu_{\tilde{P}}(x_i) - \nu_{\tilde{Q}}(x_i)|\}. \tag{23}
\]

To ensure the reasonability and validity of Equation (23), we give the following theorem.

**Theorem 1.** Let \( X = \{x_1, ..., x_n\} \) be a universe of discourses. The proposed distance \( d_{H}(\tilde{P}, \tilde{Q}) \) between PFSs \( \tilde{P} \) and \( \tilde{Q} \) should satisfy the following properties (C1)–(C5):

- (C1) (Nonnegativity) \( 0 \leq d_{H}(\tilde{P}, \tilde{Q}) \leq 1 \);
- (C2) (Separability) \( d_{H}(\tilde{P}, \tilde{Q}) = 0 \) if and only if \( \tilde{P} = \tilde{Q} \);
- (C3) (Symmetric) \( d_{H}(\tilde{P}, \tilde{Q}) = d_{H}(\tilde{Q}, \tilde{P}) \);
- (C4) (Containment) If \( \tilde{P} \subseteq \tilde{Q} \subseteq \tilde{R} \), then \( d_{H}(\tilde{P}, \tilde{Q}) \leq d_{H}(\tilde{P}, \tilde{R}) \) and \( d_{H}(\tilde{Q}, \tilde{R}) \leq d_{H}(\tilde{P}, \tilde{R}) \);
- (C5) (Triangle inequality) For any \( \tilde{P}, \tilde{Q}, \) and \( \tilde{R} \) then \( d_{H}(\tilde{P}, \tilde{Q}) \leq d_{H}(\tilde{P}, \tilde{R}) + d_{H}(\tilde{Q}, \tilde{R}) \).

**Proof.** Let \( \tilde{P} \) and \( \tilde{Q} \) be any two PFSs on the finite universe \( X = \{x_1, ..., x_n\} \) of discourses. Then, \( d_{H}(\tilde{P}, \tilde{Q}) \) of Equation (23) is positive, that is, \( d_{H}(\tilde{P}, \tilde{Q}) \geq 0 \). Since \( d_{H}(\tilde{P}, \tilde{Q}) \) is normalized, it is less than 1, that is, \( d_{H}(\tilde{P}, \tilde{Q}) \leq 1 \). The condition (C1) is proved. If \( \tilde{P} = \tilde{Q} \), then for each \( x_i \in X \), \( \mu_{\tilde{P}}^2(x_i) = \mu_{\tilde{Q}}^2(x_i) \) and \( \nu_{\tilde{P}}^2(x_i) = \nu_{\tilde{Q}}^2(x_i) \), thus, \( d_{H}(\tilde{P}, \tilde{Q}) = 0 \). Conversely, if \( d_{H}(\tilde{P}, \tilde{Q}) = 0 \), then for each \( x_i \in X \), \( \max\{|\mu_{\tilde{P}}^2(x_i) - \mu_{\tilde{Q}}^2(x_i)|, |\nu_{\tilde{P}}^2(x_i) - \nu_{\tilde{Q}}^2(x_i)|\} \) is positive, that is, \( d_{H}(\tilde{P}, \tilde{Q}) \geq 0 \). The conditions (C2)–(C5) are proved.
(x_i) - \mu_\tilde{Q}^2(x_i), |v_\tilde{P}^2(x_i) - v_\tilde{Q}^2(x_i)| = 0. Hence, |\mu_\tilde{P}^2(x_i) - \mu_\tilde{Q}^2(x_i)| = 0 and |v_\tilde{P}^2(x_i) - v_\tilde{Q}^2(x_i)| = 0, and so \tilde{P} = \tilde{Q}. Thus, the separability condition (C2) is proved. For the symmetric property (C3), we can show that \(d_H(\tilde{P}, \tilde{Q}) = d_H(\tilde{Q}, \tilde{P})\) holds because, for each \(x_i \in X\), we have \(|\mu_\tilde{P}^2(x_i) - \mu_\tilde{Q}^2(x_i)| = |\mu_\tilde{Q}^2(x_i) - \mu_\tilde{P}^2(x_i)|\) and \(|v_\tilde{P}^2(x_i) - v_\tilde{Q}^2(x_i)| = |v_\tilde{Q}^2(x_i) - v_\tilde{P}^2(x_i)|\). We next prove the containment property (C4). Let \(\tilde{P} \subseteq \tilde{Q} \subseteq \tilde{R}\), then \(\mu_\tilde{P}^2(x_i) \leq \mu_\tilde{Q}^2(x_i) \leq \mu_\tilde{R}^2(x_i)\) and \(v_\tilde{P}^2(x_i) \geq v_\tilde{Q}^2(x_i) \geq v_\tilde{R}^2(x_i), \forall x_i \in X\). Thus, we have

\[
H(I_\tilde{P}(x_i), I_\tilde{R}(x_i)) = \max\{|\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|, |v_\tilde{P}^2(x_i) - v_\tilde{R}^2(x_i)|\},
\]

\[
H(I_\tilde{Q}(x_i), I_\tilde{R}(x_i)) = \max\{|\mu_\tilde{Q}^2(x_i) - \mu_\tilde{R}^2(x_i)|, |v_\tilde{Q}^2(x_i) - v_\tilde{R}^2(x_i)|\}.
\]

(i) If \(|\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)| \geq |v_\tilde{P}^2(x_i) - v_\tilde{R}^2(x_i)|\), then \(H(I_\tilde{P}(x_i), I_\tilde{R}(x_i)) = |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\). But we have \(|v_\tilde{P}^2(x_i) - v_\tilde{R}^2(x_i)| \leq |v_\tilde{P}^2(x_i) - v_\tilde{Q}^2(x_i)| \leq |\mu_\tilde{P}^2(x_i) - \mu_\tilde{Q}^2(x_i)|\), \(|v_\tilde{Q}^2(x_i) - v_\tilde{R}^2(x_i)| \leq |v_\tilde{P}^2(x_i) - v_\tilde{R}^2(x_i)| |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\). Contrarily, we have \(|\mu_\tilde{P}^2(x_i) - \mu_\tilde{Q}^2(x_i)| \leq |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\) and \(|\mu_\tilde{Q}^2(x_i) - \mu_\tilde{R}^2(x_i)| \leq |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\). On combining the above inequalities, we obtain, \(H(I_\tilde{P}(x_i), I_\tilde{Q}(x_i)) \leq H(I_\tilde{P}(x_i), I_\tilde{R}(x_i))\) and \(H(I_\tilde{Q}(x_i), I_\tilde{R}(x_i)) \leq H(I_\tilde{P}(x_i), I_\tilde{R}(x_i))\). Hence, it follows that \(d_H(\tilde{P}, \tilde{Q}) \leq d_H(\tilde{P}, \tilde{R})\) and \(d_H(\tilde{Q}, \tilde{R}) \leq d_H(\tilde{P}, \tilde{R})\).

(ii) If \(|\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)| \leq |v_\tilde{P}^2(x_i) - v_\tilde{R}^2(x_i)|\), then \(H(I_\tilde{P}(x_i), I_\tilde{R}(x_i)) = |v_\tilde{P}^2(x_i) - v_\tilde{R}^2(x_i)|\). But we have \(|\mu_\tilde{P}^2(x_i) - \mu_\tilde{Q}^2(x_i)| \leq |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\) \(|\mu_\tilde{Q}^2(x_i) - \mu_\tilde{R}^2(x_i)| \leq |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\) \(|\mu_\tilde{Q}^2(x_i) - \mu_\tilde{R}^2(x_i)| \leq |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\). Contrarily, we have \(|v_\tilde{P}^2(x_i) - v_\tilde{Q}^2(x_i)| \leq |v_\tilde{P}^2(x_i) - v_\tilde{R}^2(x_i)|\) \(|\mu_\tilde{P}^2(x_i) - \mu_\tilde{Q}^2(x_i)| \leq |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\) \(|\mu_\tilde{Q}^2(x_i) - \mu_\tilde{R}^2(x_i)| \leq |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|\). On combining the above inequalities, we get \(H(I_\tilde{P}(x_i), I_\tilde{Q}(x_i)) \leq H(I_\tilde{P}(x_i), I_\tilde{R}(x_i))\) and \(H(I_\tilde{Q}(x_i), I_\tilde{R}(x_i)) \leq H(I_\tilde{P}(x_i), I_\tilde{R}(x_i))\). Hence, it follows that \(d_H(\tilde{P}, \tilde{Q}) \leq d_H(\tilde{P}, \tilde{R})\) and \(d_H(\tilde{Q}, \tilde{R}) \leq d_H(\tilde{P}, \tilde{R})\). Thus, the containment property (C4) is proved.

Next, we prove triangle inequality (C5). Let \(\tilde{P}, \tilde{Q}\), and \(\tilde{R}\) be any three PFSs with membership functions \(\mu_\tilde{P}^2(x_i), \mu_\tilde{Q}^2(x_i), \mu_\tilde{R}^2(x_i)\) and nonmemberships functions \(v_\tilde{P}^2(x_i), v_\tilde{Q}^2(x_i), v_\tilde{R}^2(x_i)\).

(i) If \(|\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)| \geq |v_\tilde{P}^2(x_i) - v_\tilde{R}^2(x_i)|\), then

\[
H(I_\tilde{P}(x_i), I_\tilde{R}(x_i)) = |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)|,
\]

\[
H(I_\tilde{Q}(x_i), I_\tilde{R}(x_i)) = |\mu_\tilde{P}^2(x_i) - \mu_\tilde{R}^2(x_i)| + |\mu_\tilde{Q}^2(x_i) - \mu_\tilde{R}^2(x_i)|
\]

\[
= \max\{|\mu_\tilde{P}^2(x_i) - \mu_\tilde{Q}^2(x_i)|, |v_\tilde{P}^2(x_i) - v_\tilde{Q}^2(x_i)|\}
\]

\[
+ \max\{|\mu_\tilde{Q}^2(x_i) - \mu_\tilde{R}^2(x_i)|, |v_\tilde{Q}^2(x_i) - v_\tilde{R}^2(x_i)|\}
\]

\[
= H(I_\tilde{P}(x_i), I_\tilde{Q}(x_i)) + H(I_\tilde{Q}(x_i), I_\tilde{R}(x_i)) = d_H(\tilde{P}, \tilde{Q}) + d_H(\tilde{Q}, \tilde{R}).
\]

Similarly,
(ii) If \( |\mu^2_P(x_i) - \mu^2_R(x_i)| \leq |\nu^2_P(x_i) - \nu^2_R(x_i)| \) then
\[
H(I_P(x_i), I_R(x_i)) = |\nu^2_P(x_i) - \nu^2_R(x_i)|,
\]
\[
H(I_P(x_i), I_R(x_i)) = |\nu^2_P(x_i) - \nu^2_R(x_i)| + \nu^2_Q(x_i) - \nu^2_R(x_i)|,
\]
\[
H(I_P(x_i), I_R(x_i)) \leq |\nu^2_P(x_i) - \nu^2_Q(x_i)| + |\nu^2_Q(x_i) - \nu^2_R(x_i)|
\]
\[
= \max\{|\mu^2_P(x_i) - \mu^2_Q(x_i)|, |\nu^2_P(x_i) - \nu^2_Q(x_i)|\}
\]
\[
+ \max\{|\mu^2_Q(x_i) - \mu^2_R(x_i)|, |\nu^2_Q(x_i) - \nu^2_R(x_i)|\}
\]
\[
= H(I_P(x_i), I_Q(x_i)) + H(I_Q(x_i), I_R(x_i)) = d_H(\tilde{P}, \tilde{Q}) + d_H(\tilde{Q}, \tilde{R}).
\]

It follows that \( d_H(\tilde{P}, \tilde{R}) \leq d_H(\tilde{P}, \tilde{Q}) + d_H(\tilde{Q}, \tilde{R}) \). From (i) and (ii), we prove the property (C5).  \( \Box \)

Atanassov\(^2\) and Szmidt and Kacprzyk\(^{18}\) proposed some methods for measuring the distance between IFSs, which are generalizations of Hamming distance, Euclidean distance, and their normalized versions. Grzegorzewski\(^3\) suggested another generalization of these distances based on the Hausdorff metric for IFSs. In this paper, we also consider the generalization of the distances based on the Hausdorff metric between PFSs as follows.

**Definition 5.** Let \( X = \{x_1, \ldots, x_n\} \) be the universe of discourses. For any two PFSs \( \tilde{P} = \{ (x_i, \mu^2_P(x_i), \nu^2_P(x_i)) : x_i \in X \} \) and \( \tilde{Q} = \{ (x_i, \mu^2_Q(x_i), \nu^2_Q(x_i)) : x_i \in X \} \), we propose the following measures based on the Hausdorff metric:

(i) **Hamming distance**
\[
h_{PH}(\tilde{P}, \tilde{Q}) = \sum_{i=1}^{n} \max\{|\mu^2_P(x_i) - \mu^2_Q(x_i)|, |\nu^2_P(x_i) - \nu^2_Q(x_i)|\},
\]
\[
(24)
\]

(ii) **Normalized Hamming distance**
\[
h_{nP}(\tilde{P}, \tilde{Q}) = \frac{1}{n} \sum_{i=1}^{n} \max\{|\mu^2_P(x_i) - \mu^2_Q(x_i)|, |\nu^2_P(x_i) - \nu^2_Q(x_i)|\},
\]
\[
(25)
\]

(iii) **Euclidean distance**
\[
e_{PH}(\tilde{P}, \tilde{Q}) = \sqrt{\sum_{i=1}^{n} \max\{(\mu^2_P(x_i) - \mu^2_Q(x_i))^2, (\nu^2_P(x_i) - \nu^2_Q(x_i))^2\}},
\]
\[
(26)
\]

(iv) **Normalized Euclidean distance**
\[
e_{nP}(\tilde{P}, \tilde{Q}) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \max\{(\mu^2_P(x_i) - \mu^2_Q(x_i))^2, (\nu^2_P(x_i) - \nu^2_Q(x_i))^2\}},
\]
\[
(27)
\]
Lemma 1. Let \( X = \{x_1, ..., x_n\} \) be the universe of discourses. Then, \( h_{PH}, h_{nPH}, e_{PH}, e_{nPH}, e_{gnPH} \) given by Equations (24) to (27) are metrics.

Proof. The proof is similar to Theorem 1. \(\square\)

Lemma 2. For any two PFSs \( \tilde{P} = \{(x_i, \mu_{\tilde{P}}(x_i), \nu_{\tilde{P}}(x_i)) : x_i \in X\} \) and \( \tilde{Q} = \{(x_i, \mu_{\tilde{Q}}(x_i), \nu_{\tilde{Q}}(x_i)) : x_i \in X\} \) on the universe \( X = \{x_1, x_2, ..., x_n\} \) of discourses, we obtain the following inequalities:

\[
\begin{align*}
&h_{PH}(\tilde{P}, \tilde{Q}) \leq n, \quad h_{nPH}(\tilde{P}, \tilde{Q}) \leq 1, \quad e_{PH}(\tilde{P}, \tilde{Q}) \leq \sqrt{n}, \quad e_{nPH}(\tilde{P}, \tilde{Q}) \leq 1.
\end{align*}
\]

Proof. For each \( x_i \in X \), we have \(|\mu_{\tilde{P}}^2(x_i) - \mu_{\tilde{Q}}^2(x_i)| \leq 1 \) and \(|\nu_{\tilde{P}}^2(x_i) - \nu_{\tilde{Q}}^2(x_i)| \leq 1\). Then, we have \( h_{PH}(\tilde{P}, \tilde{Q}) \leq \sum_{i=1}^{n} 1 = n \). Thus, we obtain \( h_{PH}(\tilde{P}, \tilde{Q}) \leq n \). Since \( h_{nPH}(\tilde{P}, \tilde{Q}) \leq \sqrt{\sum_{i=1}^{n} 1} = \sqrt{n} \), we have \( h_{nPH}(\tilde{P}, \tilde{Q}) \leq \sqrt{n} \). Because \( e_{PH}(\tilde{P}, \tilde{Q}) \leq \sqrt{(1/n) \sum_{i=1}^{n} 1} = 1 \), and then \( e_{nPH}(\tilde{P}, \tilde{Q}) \leq 1 \). \(\square\)

Lemma 3. For any two PFSs \( \tilde{P} = \{(x_i, \mu_{\tilde{P}}(x_i), \nu_{\tilde{P}}(x_i)) : x_i \in X\} \) and \( \tilde{Q} = \{(x_i, \mu_{\tilde{Q}}(x_i), \nu_{\tilde{Q}}(x_i)) : x_i \in X\} \) on the universe \( X = \{x_1, x_2, ..., x_n\} \) of discourses, we obtain the following inequalities:

\[
\begin{align*}
&h_p(\tilde{P}, \tilde{Q}) \leq h_{PH}(\tilde{P}, \tilde{Q}), \quad h_{np}(\tilde{P}, \tilde{Q}) \leq h_{nPH}(\tilde{P}, \tilde{Q}), \quad e_p(\tilde{P}, \tilde{Q}) \leq e_{PH}(\tilde{P}, \tilde{Q}), \quad e_{np}(\tilde{P}, \tilde{Q}) \leq e_{nPH}(\tilde{P}, \tilde{Q}).
\end{align*}
\]

Proof. We only give the proof for \( h_p(\tilde{P}, \tilde{Q}) \leq h_{PH}(\tilde{P}, \tilde{Q}) \) and leave the similar proofs of the others for the reader. For any nonnegative real numbers \( p \) and \( q \), we can get \((1/2)(p^2 + q^2) \leq \max(p^2, q^2)\). Thus, we have

\[
\begin{align*}
\frac{1}{2}(|\mu_{\tilde{P}}^2(x_i) - \mu_{\tilde{Q}}^2(x_i)| + |\nu_{\tilde{P}}^2(x_i) - \nu_{\tilde{Q}}^2(x_i)|) \\
\leq \max(|\mu_{\tilde{P}}^2(x_i) - \mu_{\tilde{Q}}^2(x_i)|, |\nu_{\tilde{P}}^2(x_i) - \nu_{\tilde{Q}}^2(x_i)|).
\end{align*}
\]

Hence, \( h_p(\tilde{P}, \tilde{Q}) \leq h_{PH}(\tilde{P}, \tilde{Q}) \). The proof is completed. \(\square\)

Lemma 4. For any two PFSs \( \tilde{P} = \{(x_i, \mu_{\tilde{P}}(x_i), 1 - \mu_{\tilde{P}}(x_i)) : x_i \in X\} \) and \( \tilde{Q} = \{(x_i, \mu_{\tilde{Q}}(x_i), 1 - \mu_{\tilde{Q}}(x_i)) : x_i \in X\} \) on the universe \( X = \{x_1, x_2, ..., x_n\} \) of discourses, we obtain the following inequalities:

\[
\begin{align*}
d_j(\tilde{P}, \tilde{Q}) \leq h_{PH}(\tilde{P}, \tilde{Q}), \quad d_{nj}(\tilde{P}, \tilde{Q}) \leq h_{nPH}(\tilde{P}, \tilde{Q}), \quad e_j(\tilde{P}, \tilde{Q}) \leq e_{PH}(\tilde{P}, \tilde{Q}), \quad e_{nj}(\tilde{P}, \tilde{Q}) \leq e_{nPH}(\tilde{P}, \tilde{Q}).
\end{align*}
\]
Proof. It is easy to be proved.

Now we use our main result, Equation (23) to establish the weighted distance in this section and different similarity measures in the next section. In applications and ranking of alternatives, a weight vector \( w \) of elements \( x \in X \) is usually considered. Therefore, we present the weighted Hausdorff distance for PFSs. Suppose that the weight of each \( x_i \in X \) is \( w_i, i = 1, ..., n \) with \( \sum_{i=1}^{n} w_i = 1 \), where \( 0 \leq w_i \leq 1 \). Then the Pythagorean weighted Hausdorff distance is defined as follows:

\[
d_{WH}(\tilde{P}, \tilde{Q}) = \sum_{i=1}^{n} w_i H(I_p(x_i), I_q(x_i)). \tag{28}
\]

Remark 1. Equation (28) will become Equation (23) if we replace \( w_i = 1/n \), for \( i = 1, 2, ..., n \). Consequently, Equation (23) is a special case of Equation (28).

Property 1. If \( d_H(\tilde{P}, \tilde{Q}) \) is the distance between two PFSs \( \tilde{P} \) and \( \tilde{Q} \), then the similarity measure \( S \) between them is denoted by \( S(\tilde{P}, \tilde{Q}) = 1 - d_H(\tilde{P}, \tilde{Q}) \).

Property 2. For any three PFSs, \( \tilde{P}, \tilde{Q}, \) and \( \tilde{R} \) on \( X = \{x_1, x_2, ..., x_n\} \) with an arbitrary length such that \( \tilde{P} \subseteq \tilde{Q} \subseteq \tilde{R} \), we have \( d_H(\tilde{P}, \tilde{R}) \geq \max\{d_H(\tilde{P}, \tilde{Q}), d_H(\tilde{Q}, \tilde{R})\} \).

Property 3. For any two PFSs \( \tilde{P} \) and \( \tilde{Q} \), the forward and backward Hausdorff distances obey either symmetric or asymmetric property, that is,

\[
d_H(\tilde{P}, \tilde{Q}) =
\begin{cases}
|\mu^2_\tilde{P}(x_i) - \mu^2_\tilde{Q}(x_i)| \text{ or } |\nu^2_\tilde{P}(x_i) - \nu^2_\tilde{Q}(x_i)| & \text{(symmetric) if } |\mu^2_\tilde{P}(x_i) - \mu^2_\tilde{Q}(x_i)| \\
|\nu^2_\tilde{P}(x_i) - \nu^2_\tilde{Q}(x_i)|, & |\mu^2_\tilde{P}(x_i) - \mu^2_\tilde{Q}(x_i)| \\
\max(|\mu^2_\tilde{P}(x_i) - \mu^2_\tilde{Q}(x_i)|, |\nu^2_\tilde{P}(x_i) - \nu^2_\tilde{Q}(x_i)|) & \text{(asymmetric) if } |\mu^2_\tilde{P}(x_i) - \mu^2_\tilde{Q}(x_i)| \\
\neq |\nu^2_\tilde{P}(x_i) - \nu^2_\tilde{Q}(x_i)|.
\end{cases}
\]

4 | SIMILARITY MEASURES FOR PFS

It is commonly known that the distance and similarity measure is a dual concept. Therefore, we may use the distance between two PFSs to define similarity between two PFSs on the basis of the Hausdorff metric. Let \( f \) be a monotone decreasing function. Since \( 0 \leq d_H(\tilde{P}, \tilde{Q}) \leq 1 \), \( f(1) \leq f(d_H(\tilde{P}, \tilde{Q})) \leq f(0) \). This implies \( 0 \leq (f(d_H(\tilde{P}, \tilde{Q})) - f(1))/(f(0) - f(1)) \leq 1 \). Hence, the similarity measure between PFSs \( \tilde{P} \) and \( \tilde{Q} \) can be defined as follows:

Definition 6. Let the universe of discourse be \( X = \{x_1, x_2, ..., x_n\} \) and \( \tilde{P} = \{\langle x, \mu_\tilde{P}(x), \nu_\tilde{P}(x) \rangle : x \in X \} \) and \( \tilde{Q} = \{\langle x, \mu_\tilde{Q}(x), \nu_\tilde{Q}(x) \rangle : x \in X \} \) be two PFSs on \( X \). Let \( f \) be a monotone decreasing function. Then, the similarity measure between \( \tilde{P} \) and \( \tilde{Q} \) can be defined as follows:
By using Equation (29), different kinds of similarity measures can be obtained by choosing an appropriate and reasonable $f$. The most simplest linear function $f$ may be chosen as $f(x) = 1 - x$, and then the similarity measure between PFSs $\tilde{P}$ and $\tilde{Q}$ by using Equation (23) can be represented as follows:

$$S_{L}(\tilde{P}, \tilde{Q}) = 1 - d_{H}(\tilde{P}, \tilde{Q}).$$

Again, we may also choose another suitable function as a simple rational function $f(x) = \frac{1}{1 + x}$. Then the similarity measure between PFSs $\tilde{P}$ and $\tilde{Q}$ is defined as follows:

$$S_{Q}(\tilde{P}, \tilde{Q}) = \frac{1 - d_{H}(\tilde{P}, \tilde{Q})}{1 + d_{H}(\tilde{P}, \tilde{Q})}.$$

Now, we consider another well-known function, an exponential function $f(x) = e^{-x}$, therefore, we construct another similarity measure between PFSs $\tilde{P}$ and $\tilde{Q}$ by using the exponential function as follows:

$$S_{E}(\tilde{P}, \tilde{Q}) = e^{-d_{H}(\tilde{P}, \tilde{Q})} - e^{-1}.$$

Remark 2. One can also write the similarities by using the proposed distances Equations (13) to (20) and Equations (24) to (27).

Considering the continuous universe of discourse $X = [a, b]$, we can get the following analogous results. For any two PFSs $\tilde{P} = \{\langle x, \mu_{P}(x), \nu_{P}(x) \rangle: x \in X \}$ and $\tilde{Q} = \{\langle x, \mu_{Q}(x), \nu_{Q}(x) \rangle: x \in X \}$, we define the distance measure $d_{PHC}$ as follows:

$$d_{PHC}(\tilde{P}, \tilde{Q}) = \frac{1}{b - a} \int_{a}^{b} H(I_{P}(x_{i}), I_{Q}(x_{i}))dx,$$

where $I_{P}(x_{i}) = [\mu_{P}^{2}(x_{i}), 1 - \nu_{P}^{2}(x_{i})]$ and $I_{Q}(x_{i}) = [\mu_{Q}^{2}(x_{i}), 1 - \nu_{Q}^{2}(x_{i})]$, $x \in [a, b]$. In general, each element in the universe may have a different importance. We need to consider a weight vector $w$ for the element $x \in X$. Likewise, in the continuous case, we suppose that the weight of $X = [a, b]$, $\forall x \in X$ is $w(x)$, where $0 \leq w(x) \leq 1$ and $\int_{a}^{b} w(x) = 1$. Then the weighted Hausdorff distance between PFSs $\tilde{P}$ and $\tilde{Q}$ is defined as

$$d_{PHwC}(\tilde{P}, \tilde{Q}) = \int_{a}^{b} w(x)H(I_{P}(x_{i}), I_{Q}(x_{i}))dx.$$

Remark 3. Obviously, Equation (34) becomes Equation (33) if we have $w(x) = 1/(b - a)$, $\forall x \in [a, b]$. Therefore, Equation (33) is a special case of Equation (34). Furthermore, we
can obtain similarity measures between \( \tilde{P} \) and \( \tilde{Q} \) by replacing \( d_H(\tilde{P}, \tilde{Q}) \) in Equation (29) with \( d_{WH}(\tilde{P}, \tilde{Q}), d_{PHC}(\tilde{P}, \tilde{Q}) \), or \( d_{PHW}(\tilde{P}, \tilde{Q}) \).

5 \hspace{1cm} \textbf{NUMERICAL EXAMPLES AND COMPARISONS}

In this section, we present some examples to exhibit the validity, suitability and applicability of our proposed distances and similarity measures between PFSs based on the Hausdorff metric.

\textbf{Example 1.} Let \( X = \{x_1, x_2, x_3\} \) be the universe of discourse and suppose that there are patterns consisting of PFSs as follows:

\[ \tilde{P}_1 = \{\langle x_1, 0.9, 0.3 \rangle, \langle x_2, 0.8, 0.5 \rangle, \langle x_3, 0.7, 0.6 \rangle\}; \]

\[ \tilde{P}_2 = \{\langle x_1, 0.7, 0.4 \rangle, \langle x_2, 0.6, 0.5 \rangle, \langle x_3, 0.6, 0.3 \rangle\}. \]

Assume that a sample \( \tilde{Q} = \{\langle x_1, 0.9, 0.3 \rangle, \langle x_2, 0.8, 0.5 \rangle, \langle x_3, 0.6, 0.3 \rangle\} \) is given. By using Equations (13) to (16), we have the following distance measures:

\[ h_p(\tilde{P}_1, \tilde{Q}) = 0.2000, \quad h_{np}(\tilde{P}_1, \tilde{Q}) = 0.0667, \]

\[ e_p(\tilde{P}_1, \tilde{Q}) = 0.2119, \quad e_{np}(\tilde{P}_1, \tilde{Q}) = 0.1223; \]

\[ h_p(\tilde{P}_2, \tilde{Q}) = 0.3350, \quad h_{np}(\tilde{P}_2, \tilde{Q}) = 0.1117, \]

\[ e_p(\tilde{P}_2, \tilde{Q}) = 0.3047, \quad e_{np}(\tilde{P}_2, \tilde{Q}) = 0.1759. \]

The above results clearly indicate that the sample \( \tilde{Q} \) is similar to the pattern \( \tilde{P}_1 \) according to the principle of the minimum degree of dissimilarity. The results are also according to our intuition and expectations.

Now we consider the three PFSs viz. the membership degree \( \mu(x) \), the nonmembership degree \( \nu(x) \) and the intuitionistic fuzzy index \( \pi(x) \) with

\[ \tilde{P}_1 = \{\langle x_1, 0.9, 0.3, 0.3162 \rangle, \langle x_2, 0.8, 0.5, 0.3317 \rangle, \langle x_3, 0.7, 0.6, 0.3873 \rangle\}; \]

\[ \tilde{P}_2 = \{\langle x_1, 0.7, 0.4, 0.5916 \rangle, \langle x_2, 0.6, 0.5, 0.6245 \rangle, \langle x_3, 0.6, 0.3, 0.7416 \rangle\}. \]

Assume that a sample \( \tilde{Q} = \{\langle x_1, 0.9, 0.3, 0.3162 \rangle, \langle x_2, 0.8, 0.5, 0.3317 \rangle, \langle x_3, 0.6, 0.3, 0.7416 \rangle\} \) is given. Then, we have the following results by using Equations (17) to (20):

\[ \widehat{h}_p(\tilde{P}_1, \tilde{Q}) = 0.4000, \quad \widehat{h}_{np}(\tilde{P}_1, \tilde{Q}) = 0.1333, \quad \widehat{e}_p(\tilde{P}_1, \tilde{Q}) = 0.3534, \]

\[ \widehat{e}_{np}(\tilde{P}_1, \tilde{Q}) = 0.2040; \]

\[ \widehat{h}_p(\tilde{P}_2, \tilde{Q}) = 0.6000, \quad \widehat{h}_{np}(\tilde{P}_2, \tilde{Q}) = 0.2000, \quad \widehat{e}_p(\tilde{P}_2, \tilde{Q}) = 0.4041, \]

\[ \widehat{e}_{np}(\tilde{P}_2, \tilde{Q}) = 0.2333. \]

It is seen that the values get increasing, but the results remain the same, indicating that the sample \( \tilde{Q} \) is similar to the pattern \( \tilde{P}_1 \) according to the principle of minimum degree of dissimilarity. Now, the results of our proposed distance measures Equation (23) are as follows:

\[ d_H(\tilde{P}_1, \tilde{Q}) = 0.0900, \quad d_H(\tilde{P}_2, \tilde{Q}) = 0.2000. \]

Based on the above numerical results, the sample \( \tilde{Q} \) is similar to the pattern \( \tilde{P}_1 \) according to the principle of minimum degree of dissimilarity. The pattern recognition results are the same as earlier, but we can see a considerable decrease in numerical values.
The following are the results of the other proposed measures Equations (24) to (27) of PFSs based on the Hausdorff metric:

\[ h_{PH}(\tilde{P}_1, \tilde{Q}) = 0.2700, \quad h_{nPH}(\tilde{P}_1, \tilde{Q}) = 0.0900, \quad e_{PH}(\tilde{P}_1, \tilde{Q}) = 0.2700, \]
\[ e_{nPH}(\tilde{P}_1, \tilde{Q}) = 0.1559; \]
\[ h_{PH}(\tilde{P}_2, \tilde{Q}) = 0.6000, \quad h_{nPH}(\tilde{P}_2, \tilde{Q}) = 0.2000, \quad e_{PH}(\tilde{P}_2, \tilde{Q}) = 0.4252, \]
\[ e_{nPH}(\tilde{P}_2, \tilde{Q}) = 0.2455. \]

Obviously, the sample \( \tilde{Q} \) is similar to the pattern \( \tilde{P}_1 \) according to the principle of minimum degree of dissimilarity, but there are some changes in numerical values. Further, we observe that the following results are analogous to each other:

\[ d_{PH}(\tilde{P}_1, \tilde{Q}) = 0.0900 = d_{nPH}(\tilde{P}_1, \tilde{Q}) \quad \text{and} \quad d_{nPH}(\tilde{P}_2, \tilde{Q}) = 0.2000 = d_{PH}(\tilde{P}_2, \tilde{Q}). \]

To further check the validity and applicability of the proposed similarity measures Equations (30) to (32) in pattern recognition, we next exhibit the following numerical examples.

**Example 2.** Suppose that there are three patterns denoted by PFSs in the universe of discourse \( X = \{x_1, x_2, x_3\} \). The three patterns are represented by \( \tilde{P}_1 = \{(x_1, 0.7, 0.7), (x_2, 0.6, 0.6), (x_3, 0.5, 0.5)\} \); \( \tilde{P}_2 = \{(x_1, 0.7, 0.7), (x_2, 0.7, 0.7), (x_3, 0.7, 0.7)\}; \) \( \tilde{P}_3 = \{(x_1, 0.3, 0.3), (x_2, 0.3, 0.3), (x_3, 0.3, 0.3)\} \). Assume that a sample \( \tilde{Q} = \{(x_1, 0.7, 0.7), (x_2, 0.6, 0.6), (x_3, 0.5, 0.5)\} \) is given. By using Equations (30) to (32), we have \( S_L(\tilde{P}_1, \tilde{Q}) = 1 \), \( S_Q(\tilde{P}_1, \tilde{Q}) = 1 \), \( S_E(\tilde{P}_1, \tilde{Q}) = 1 \); \( S_L(\tilde{P}_2, \tilde{Q}) = 0.8767 \), \( S_Q(\tilde{P}_2, \tilde{Q}) = 0.7805 \), \( S_E(\tilde{P}_2, \tilde{Q}) = 0.8165 \); \( S_L(\tilde{P}_3, \tilde{Q}) = 0.7233 \), \( S_Q(\tilde{P}_3, \tilde{Q}) = 0.5665 \), \( S_E(\tilde{P}_3, \tilde{Q}) = 0.6176 \). From the above results using the proposed similarity measures Equations (30) to (32), it is obvious that \( \tilde{P}_1 = \tilde{Q} \). Furthermore, we can see that the sample \( \tilde{Q} \) is more similar to the pattern \( \tilde{P}_2 \) than the pattern \( \tilde{P}_3 \). Hence, according to principle of the maximum degree of similarity between PFSs, the sample \( \tilde{Q} \) is identical to the pattern \( \tilde{P}_1 \).

**Example 3.** Let \( X = \{x_1, x_2, x_3\} \) be the universe of discourse and let the two patterns consist of PFSs with \( \tilde{P}_1 = \{(x_1, 0.4, 0.4), (x_2, 0.4, 0.4), (x_3, 0.4, 0.4)\} \) and \( \tilde{P}_2 = \{(x_1, 0.6, 0.6), (x_2, 0.6, 0.6), (x_3, 0.6, 0.6)\}. \) Assume that a sample \( \tilde{Q} = \{(x_1, 0.5, 0.5), (x_2, 0.5, 0.5), (x_3, 0.3, 0.3)\} \) is given. By using Equations (30) to (32), we have \( S_L(\tilde{P}_1, \tilde{Q}) = 0.9100 \), \( S_Q(\tilde{P}_1, \tilde{Q}) = 0.8349 \), \( S_E(\tilde{P}_1, \tilde{Q}) = 0.8638 \); \( S_L(\tilde{P}_2, \tilde{Q}) = 0.8367 \), \( S_Q(\tilde{P}_2, \tilde{Q}) = 0.7192 \), \( S_E(\tilde{P}_2, \tilde{Q}) = 0.7617 \). Based on above results, it can be seen that the sample \( \tilde{Q} \) is more similar to the pattern \( \tilde{P}_2 \) than the pattern \( \tilde{P}_1 \) Therefore, it is evident that the sample \( \tilde{Q} \) is similar to the pattern \( \tilde{P}_3 \) according to the principle of the maximum degree of similarity between PFSs.

**Example 4.** Assume that there are three patterns denoted by PFSs in \( X = \{x_1, x_2, x_3\} \) with \( \tilde{P}_1 = \{(x_1, 0.6, 0.5), (x_2, 0.3, 0.7), (x_3, 0.5, 0.8)\}; \) \( \tilde{P}_2 = \{(x_1, 0.7, 0.6), (x_2, 0.9, 0.4), (x_3, 0.4, 0.6)\}; \) and \( \tilde{P}_3 = \{(x_1, 0.6, 0.6), (x_2, 0.8, 0.4), (x_3, 0.1, 0.6)\} \). Assume that a sample \( \tilde{Q} = \{(x_1, 0.7, 0.7), (x_2, 0.8, 0.3), (x_3, 0.0, 0.6)\} \) is given. Then, we obtain \( S_L(\tilde{P}_1, \tilde{Q}) = 0.4742 \), \( S_Q(\tilde{P}_1, \tilde{Q}) = 0.5254 \); \( S_L(\tilde{P}_2, \tilde{Q}) = 0.8467 \), \( S_Q(\tilde{P}_2, \tilde{Q}) = 0.7342 \), \( S_E(\tilde{P}_2, \tilde{Q}) = 0.7752 \); \( S_L(\tilde{P}_3, \tilde{Q}) = 0.9300 \), \( S_Q(\tilde{P}_3, \tilde{Q}) = 0.8692 \), \( S_E(\tilde{P}_3, \tilde{Q}) = 0.8930 \).
According to the above results using the similarity measures of Equations (30) to (32) between PFSs, we have that the sample \( \tilde{Q} \) is closed to the pattern \( \tilde{P}_3 \) as compared to the patterns \( \tilde{P}_1 \) and \( \tilde{P}_2 \). Thus, according to the principle of the maximum degree of similarity between PFSs, we have that the sample \( \tilde{Q} \) is similar to the pattern \( \tilde{P}_3 \).

We next give an example pertaining to characterization of the similarities between linguistic variables by using the proposed similarity measures of Equations (30) to (32). In the literature, the usage of fuzzy sets for querying a regular database was suggested by Tahani\textsuperscript{33} and database queries with fuzzy linguistic quantifiers were introduced by Kacprzyk and Ziolkowski\textsuperscript{34} and Petry.\textsuperscript{35} For querying a database, similarity measures are important techniques (see Candan et al.\textsuperscript{36}). To query a database with fuzzy queries more amicably, it is essential to clarify a degree of similarity measures between these fuzzy sets. In the next example, we characterize the similarity between linguistic variables by using our proposed similarity measures of Equations (30) to (32) between PFSs.

**Example 5.** Suppose that a PFS on \( X \) is given by \( P = \{ (x, \mu_p(x), \nu_p(x)) : x \in X \} \). For any positive number \( n \), we have the PFS \( P^n \) from Definition 3 with \( \tilde{P}^n = \{ (x, (\mu_p(x))^n, \sqrt[n]{1 - (1 - \nu_p(x))^n}) : x \in X \} \). Furthermore, we have two operators, concentration of \( P \): \( \text{CON}(P) = P^2 \), and Dilation of \( P \): \( \text{DIL}(P) = P^{1/2} \). The operations such as \( \text{CON}(P) \) and \( \text{DIL}(P) \) may be regarded as “very (\( P \))” and “more or less (\( P \)),” respectively. Now, we consider the PFS \( P \) in \( X = \{1, 2, 3, 4, 5\} \) with \( P = \{ (1, 0.3, 0.9), (2, 0.4, 0.6), (3, 0.8, 0.5), (4, 0.9, 0.2), (5, 1.0, 0.0) \} \). The PFS \( P \) can be considered as corresponding to linguistic hedges, such as \( P \) regarded as “LARGE” in \( X \). In this sense, the operations \( \text{CON}(P) \) and \( \text{DIL}(P) \) can be used as hedges, such as “More or less LARGE,” “Very LARGE,” “Very very LARGE.” Thus, we have

\[ P^{1/2} \text{ is regarded as “More or less LARGE,”} \]
\[ P^2 \text{ is regarded as “Very LARGE,”} \]
\[ P^4 \text{ is regarded as “Very very LARGE.”} \]

We use the abbreviations L for LARGE, MLL for More or less LARGE, VL for Very LARGE, and VVL for Very very LARGE. The proposed similarity measures of Equations (30) to (32) are used to calculate the degrees of similarity between PFSs. The results are shown in Table 1.

From Table 1, we obtain the following prerequisites according to the degrees of similarity between PFSs:

\[
S(L, VL) > S(L, MLL) > S(L, VVL), \quad S(LL, L) > S(MLL, VL) > S(MLL, VLL), \\
\]

From the above results, we can see that the proposed similarity measures satisfy all the requirements and actually give good ordering between L, MLL, VL, and VVL. Consequently, the proposed similarity measures of Equations (30) to (32) are valid, well suited, and applicable in a compound linguistic variable environment.
Decision making is a sort of daily activity. In the decision-making process, we select the best option or options from the available finite number of feasible options. It plays an essential role in a variety of fields such as business, biological sciences, computer science, engineering, economics, finance, management, medicines, social, and political sciences. MCDM is a process to select an optimal option from a finite number of feasible options under several criteria. To obtain the most preferred option, decision makers provide their preference information among the available options.

Imprecision is a real truth in daily life that requires close attention in matters of management and decisions. In the real-life setting with decision making process, the information available is often uncertain, vague or imprecise. PFSs are found to be a good tool to solve decision-making problems involving uncertain, vague or imprecise information with high precision. In the literature, various theories, applications, and methods based on PFSs were proposed to make MCDM more effective and accurate. In this section, we apply the proposed distances to MCDM problems. TOPSIS is an approach to identify an alternative that is closest to the positive ideal solution and farthest to the negative ideal solution in the multicriterion decision environment. We extend the TOPSIS concept to construct an algorithm for Pythagorean fuzzy TOPSIS to solve MCDM problems based on the proposed Hausdorff measures for PFSs.

MCDM problems are normally defined on a set of alternatives. Among these alternatives, decision makers have to select the best alternatives according to some criteria. Assume that there exists an alternative set $A = \{A_1, A_2, ..., A_m\}$, which consists of $m$ alternatives. Decision makers should choose the best alternative from the above set $A$ according to the set of $n$ criteria $C = \{C_1, C_2, ..., C_n\}$ with weight vector $w_i (i = 1, ..., n)$ such that $\sum_{i=1}^{n} w_i = 1$, where $0 \leq w_i \leq 1$.

Suppose that $D = (d_{ij})_{m \times n}$ is the Pythagorean fuzzy decision matrix, where $d_{ij} = (\mu_{ij}, \nu_{ij})$, $i = \{1, 2, ..., m\}$, and $j = \{1, 2, ..., n\}$ is a criteria value given by the decision maker, such that $\mu_{ij}$ indicates the degree to which the alternative $A_i$ satisfies the criteria $C_j$, and $\nu_{ij}$ denotes the...
degree to which the alternative \( A_i \) does not satisfy the criteria \( C_j \), such that 
\[ 0 \leq \mu_i^j(x) + \nu_i^j(x) \leq 1, \] 
where the function \( \mu_i^j(x) \in [0, 1] \), \( \nu_i^j(x) \in [0, 1] \), \( i = \{1, 2, ..., m\} \), and \( j = \{1, 2, ..., n\} \). A new Pythagorean fuzzy TOPSIS based on the proposed Hausdorff distance between PFSs is proposed. The algorithm with the new Pythagorean fuzzy TOPSIS for solving MCDM problems is presented as follows:

### 6.1 Step 1: Construction of Pythagorean fuzzy decision matrix and characteristic sets

In this step, we denote the evaluation values of alternatives \( A = \{A_1, A_2, ..., A_m\} \), \( i = \{1, 2, ..., m\} \) with respect to the criteria \( C = \{C_1, C_2, ..., C_n\} \), \( j = \{1, 2, ..., n\} \), by \( A_i(C_j) = P(\mu_{ij}, \nu_{ij}) \). Then the Pythagorean fuzzy decision matrix can be denoted by \( D = (A_i(C_j))_{m \times n} \). Here, \( P(\mu_{ij}, \nu_{ij}) \) represents PFN, which denotes the evaluation values given by decision makers so that the alternative \( A_i \) satisfies the criteria \( C_j \). Based on the matrix \( D = (A_i(C_j))_{m \times n} \), we specify the alternatives \( A_i \) by the characteristic sets: \( A_i = \{(C_j, (\mu_{ij}, \nu_{ij})) : C_j \in C\}, i = 1, 2, ..., m \) and \( j = 1, 2, ..., n \).

### 6.2 Step 2: New Pythagorean fuzzy positive ideal solution and Pythagorean fuzzy negative ideal solution

In the TOPSIS method, the evaluation criteria can be categorized into two categories, benefit and costs. Let \( M \) be a collection of benefit criteria and \( N \) be a collection of cost criteria. The Pythagorean fuzzy positive ideal solution (PFPIS) \( S^+ \) and the Pythagorean fuzzy negative ideal solution (PFNIS) \( S^- \) can be defined as follows:

\[
S^+ = \left[ \{C_j, ((\max_i \mu_{ij}(C_j) : j \in M), (\min_i \mu_{ij}(C_j) : j \in N)), (\min_i \nu_{ij}(C_j) : j \in M), (\max_i \nu_{ij}(C_j) : j \in N) \} : i \in m \right],
\]

\[
S^- = \left[ \{C_j, (min \mu_{ij}(C_j) : j \in M), (max \mu_{ij}(C_j) : j \in N)), (max \nu_{ij}(C_j) : j \in M), (min \nu_{ij}(C_j) : j \in N) \} : i \in m \right].
\]

### 6.3 Step 3: Construction of distance between alternative \( A_i \) and PFPIS and PFNIS

To measure the distance of each alternative \( A_i \) from PFPIS and PFNIS, the proposed Pythagorean fuzzy weighted Hausdorff distance (28) is used under assumption that weights \( w_i = 1, 2, ..., n \) with \( \sum_{i=1}^{n} w_i = 1 \) and \( 0 \leq w_i \leq 1 \). Then, we give the following expressions:

\[
T^+(A_i) = d_{WH}(A_i, S^+), \text{ and } T^-(A_i) = d_{WH}(A_i, S^-), \quad i = \{1, 2, ..., m\}.
\]

### 6.4 Step 4: Construction of degree of relative closeness

The greater value of relative closeness degree shows that an alternative is closer to PFPIS, but is away from PFNIS. The relative closeness degree \( T(A_i) \) of each alternative with respect to Pythagorean fuzzy ideal solutions is computed by the following expression:
Finally, the alternatives are ranked in preference order according to the degree of closeness.

6.5 | Step 5: Ranking of the alternatives

Ranking of all alternatives can be done according to the descending order of relative coefficient values. The alternative carrying the highest value will be considered as the most preferred alternative.

To demonstrate the feasibility and practical applicability of the above proposed method, we present the following example related to social sector.

**Example 6** (Application to social sector). In many countries, nowadays, the competition among private school services is increasing and it is much more difficult for families to choose a suitable and appropriate school among those better schools with quality education and bright future for their children. Even in communities where there are good alternatives of public schools, families often get themselves prepared to consider private sector for one or the other reasons. When trying to decide a suitable school for their children, most families found themselves confused to choose an appropriate and reasonable school among several top and renowned private schools in private sector. To cope with and overcome this important social issue, a team of veteran educationists and experts decides to rank private schools in the private sector according to a set of criteria based on their research. Suppose that families have to select the best school to equip their children with modern, up to date, and quality education. There are five possible alternatives: (a) private sector school S1, (b) private sector school S2, (c) private sector school S3, (d) private sector school S4, and (e) private sector school S5. Based on the research conducted by the veteran educationists and experts in this related field, five major criteria are identified to evaluate these five schools. These criteria are: individual attention $C_1$, academic excellence $C_2$, highly qualified and up-to-date teachers $C_3$, parents and community involvement $C_4$, and support facilities $C_5$. A detailed description of the criteria is given in Table 2. The assessments of the alternatives $A_i$ under the criteria $C_j$ are displayed in Table 3 in the form of Pythagorean fuzzy decision matrix $D = (A_i(C_j))_{m \times n}$. By using the Pythagorean fuzzy decision matrix in Table 3, we have characteristic sets of the alternatives $A_i$, PFPIS $S^+$, and PFNIS $S^-$ as follows:

$$A_1 = \{(0.7, 0.6), (0.9, 0.4), (0.8, 0.4), (0.6, 0.5), (0.5, 0.4)\},$$

$$A_2 = \{(0.6, 0.5), (0.8, 0.3), (0.9, 0.1), (0.7, 0.6), (0.5, 0.6)\},$$

$$A_3 = \{(0.8, 0.4), (0.7, 0.5), (0.6, 0.2), (0.4, 0.3), (0.7, 0.2)\},$$

$$A_4 = \{(0.6, 0.4), (0.9, 0.2), (0.8, 0.5), (0.7, 0.5), (0.8, 0.3)\},$$

$$A_5 = \{(0.8, 0.5), (0.7, 0.4), (0.9, 0.3), (0.4, 0.5), (0.6, 0.5)\}.$$

$$S^+ = \{(0.8, 0.4), (0.9, 0.2), (0.9, 0.1), (0.7, 0.3), (0.8, 0.2)\},$$

$$S^- = \{(0.6, 0.6), (0.7, 0.5), (0.6, 0.5), (0.4, 0.6), (0.5, 0.6)\}.$$

We use the weights of $w = (w_1, w_2, w_3, w_4, w_5)^T = (0.18, 0.32, 0.23, 0.13, 0.14)^T$. We now use the proposed Hausdorff weighted distance (28) to calculate the distance from each
alternative $A_i$ to the PFPI$S^+$ with $d_{WH}(A_i, S^+)$, and to the PFNi$S^-$ with $d_{WH}(A_i, S^-)$, as shown in Table 4, in the form of so-called Pythagorean separation measures. Then, the relative closeness degrees of each alternative with respect to Pythagorean fuzzy ideal solutions computed by Equation (35) are shown in Table 5. The ranking of the alternatives $A_i$ is ordered according to descending order to the relative closeness degree, as shown in Table 6. From Table 6, the largest value of $A_4$ is chosen as the best alternative. According to these $A_i$, $S^+$, and $S^-$, we can see that the chosen alternative $A_4$ is actually the most closest to positive ideal solution, but farthest from the negative ideal solution.

### 7 CONCLUSIONS

Many attempts have been made in the literature about improvement of information measures. However, there is still room to improve or present these measures in a better way and then use them to find new applications with novel directions. In this paper, we proposed several new distance and similarity measures between PFSs based on the Hausdorff metric. We first considered the Hausdorff metric for calculating the distance between PFSs. Then, we utilized this distance to create new similarity measures to calculate the degrees of similarities between PFSs. We had considered distance and similarity between PFSs on finite universes of discourses,
which are not only used in purposes of computing environment, but also in more general cases for large universal sets. Several examples, including some in pattern recognition and some in linguistic variables, were presented to make sure of the suitability, reliability and validity of our proposed methods. We then applied our proposed methods in multicriteria decision making with a Pythagorean fuzzy TOPSIS construction by presenting a practical application to social sector. On the basis of the computational results, it was seen that the proposed methods by employing the Hausdorff metric in measuring the distance and similarity for PFSs is reasonable, intuitive and well-suited in handling different kinds of applications, such as pattern recognition, linguistic variable and multicriteria decision making.

REFERENCES


**TABLE 4** Pythagorean separation measures

<table>
<thead>
<tr>
<th>$d_{WH}$</th>
<th>$S^+$</th>
<th>$S^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.1889</td>
<td>0.2442</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.1945</td>
<td>0.2174</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.2698</td>
<td>0.1786</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.1334</td>
<td>0.3003</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.2191</td>
<td>0.2124</td>
</tr>
</tbody>
</table>

**TABLE 5** Relative closeness degree

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closeness degree</td>
<td>0.5638</td>
<td>0.5278</td>
<td>0.3983</td>
<td>0.6924</td>
<td>0.4922</td>
</tr>
</tbody>
</table>

**TABLE 6** Ranking order of alternatives

<table>
<thead>
<tr>
<th>Distance</th>
<th>Ranking</th>
<th>The best alternative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{WH}$</td>
<td>$A_4$ $A_3$ $A_5$ $A_3$</td>
<td>$A_4$</td>
</tr>
</tbody>
</table>

which are not only used in purposes of computing environment, but also in more general cases for large universal sets. Several examples, including some in pattern recognition and some in linguistic variables, were presented to make sure of the suitability, reliability and validity of our proposed methods. We then applied our proposed methods in multicriteria decision making with a Pythagorean fuzzy TOPSIS construction by presenting a practical application to social sector. On the basis of the computational results, it was seen that the proposed methods by employing the Hausdorff metric in measuring the distance and similarity for PFSs is reasonable, intuitive and well-suited in handling different kinds of applications, such as pattern recognition, linguistic variable and multicriteria decision making.

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